HOSSAM GHANEM

(16) 7.7 Indeterminate Forms and I 'Hopital's Rule(B)

Using Logarithmic To Find Limits:

$\ln \lim_{x\to a} f(x) = \lim_{x\to a} \ln f(x)$		$IF \quad \ln \lim_{x\to a} f(x) = L \quad \Rightarrow \quad \lim_{x\to a} f(x) = e^L$	
$e^{\ln a} = a$	$e^0 = 1$	$e^\infty = \infty$	$e^{-\infty} = 0$

Indeterminate form:

Form		Guidelines	Example
0^0 ∞^0 1^∞	1	$L = \lim_{x\to a}[f(x)]^{g(x)} \to \begin{cases} 0^0 \\ \infty^0 \\ 1^\infty \end{cases}$	$L = \lim_{x\to 0^+} x^{x^2} \to 0^0$
	2	$\ln L = \lim_{x\to a} \ln[f(x)]^{g(x)}$ $= \lim_{x\to a} g(x) \ln[f(x)]$	$\ln L = \lim_{x\to 0^+} \ln x^{x^2}$ $= \lim_{x\to 0} x^2 \ln x$
	3	$\ln L = \lim_{x\to a} \dfrac{\ln f(x)}{\frac{1}{g(x)}} \to \begin{cases} \frac{0}{0} \\ \frac{\pm\infty}{\pm\infty} \end{cases}$	$\ln L = \lim_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x^2}} \to \dfrac{-\infty}{\infty}$
	4	$\ln L = \lim_{x\to a} \dfrac{D_x \ln f(x)}{D_x\left(\frac{1}{g(x)}\right)} = k$	$\ln L = \lim_{x\to 0^+} \dfrac{\frac{1}{x}}{\frac{-2}{x^3}} = \frac{1}{x} \cdot \frac{-x^3}{2} = -\frac{x^2}{2} = 0$
	5	$L = e^k$	$L = e^0 = 1$

$\infty - \infty$	حاول وضع المطلوب على صورة بسط ومقام وذلك بتوحيد المقامات و الاختصار أو الضرب في المرافق حتى تحصل على إحدى الصور السابقة $\frac{0}{0}$ أو $\frac{\pm\infty}{\pm\infty}$

Example 1

Find the following limits $\lim_{x\to 0^+}(1+\sin 2x)^{\csc 3x}$ 13 March 2001 A

Solution

$$L=\lim_{x\to 0^+}(1+\sin 2x)^{\csc 3x}=1^\infty$$

$$\ln L=\lim_{x\to 0^+}\csc 3x\,\ln(1+\sin 2x)=\lim_{x\to 0^+}\frac{\ln(1+\sin 2x)}{\sin 3x}=\frac{0}{0}$$

$$\overset{LR}{=}\lim_{x\to 0^+}\frac{\frac{2\cos 2x}{1+\sin 2x}}{3\cos 3x}=\lim_{x\to 0^+}\frac{2\cos 2x}{3\cos 3x\,(1+\sin 2x)}=\frac{2}{3(1+0)}=\frac{2}{3}$$

$$L=e^{\frac{2}{3}}$$

Example 2

35 2 November 2011

Find the limit if it exists $\lim_{x\to\infty}\left(\frac{x+3}{\cosh x}\right)^{1/x}$. (4 pts)

Solution

$$c\lim_{x\to\infty}\left(\frac{x+3}{\cosh x}\right)^{1/x}=0^0$$

$$\ln L=\lim_{x\to\infty}\frac{1}{x}\ln\left(\frac{x+3}{\cosh x}\right)=\lim_{x\to\infty}\frac{\ln(x+3)-\ln(\cosh x)}{x}=\lim_{x\to\infty}\frac{\ln(x+3)}{x}-\lim_{x\to\infty}\frac{\ln(\cosh x)}{x}$$

$$L_1=\lim_{x\to\infty}\frac{\ln(x+3)}{x}=\frac{\infty}{\infty}$$

$$\overset{LR}{=}\lim_{x\to\infty}\frac{\frac{1}{x+3}}{1}=0$$

$$L_2=\lim_{x\to\infty}\frac{\ln(\cosh x)}{x}=\frac{\infty}{\infty}$$

$$\overset{LR}{=}\lim_{x\to\infty}\frac{\frac{\sinh x}{\cosh x}}{1}=\lim_{x\to\infty}\tanh x=1$$

$$\ln L=L_1-L_2=0-1=-1$$

$$L=e^{-1}=\frac{1}{e}$$

Example 3

Show that $\lim_{x\to 0^+}\left(\frac{5^x+7^x}{2}\right)^{\frac{1}{x}}=\sqrt{35}$ 16 December 1999 A

Solution

$$L=\lim_{x\to 0^+}\left(\frac{5^x+7^x}{2}\right)^{\frac{1}{x}}$$

$$\ln L=\lim_{x\to 0^+}\frac{\ln\left(\frac{5^x+7^x}{2}\right)}{x}=\frac{0}{0}$$

$$\overset{LR}{=}\lim_{x\to 0^+}\frac{\frac{2}{5^x+7^x}\left(\frac{5^x\ln 5+7^x\ln 7}{2}\right)}{1}=\lim_{x\to 0^+}\frac{5^x\ln 5+7^x\ln 7}{5^x+7^x}=\frac{\ln 5+\ln 7}{2}=\frac{1}{2}\ln 35=\ln\sqrt{35}$$

$$L=e^{\ln\sqrt{35}}=\sqrt{35}$$

Example 4 Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{2\sqrt{x}}\right)^{\sqrt{x}}$ 17 May 2000 A

Solution

$$L=\lim_{x\to\infty}\left(1+\frac{1}{2\sqrt{x}}\right)^{\sqrt{x}}=\lim_{x\to\infty}\left(1+\frac{1}{2}x^{-\frac{1}{2}}\right)^{x^{\frac{1}{2}}}=1^{\infty}$$

$$\ln L=\lim_{x\to\infty}x^{\frac{1}{2}}\ln\left(1+\frac{1}{2}x^{-\frac{1}{2}}\right)=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{2}x^{-\frac{1}{2}}\right)}{x^{-\frac{1}{2}}}=\frac{0}{0}$$

$$\overset{LR}{=}\lim_{x\to\infty}\frac{-\frac{1}{4}x^{-\frac{3}{2}}\left(\frac{1}{1+\frac{1}{2}x^{-\frac{1}{2}}}\right)}{-\frac{1}{2}\,x^{-\frac{3}{2}}}=\lim_{x\to\infty}\frac{-1}{-2\left(1+\frac{1}{2}x^{-\frac{1}{2}}\right)}=\frac{1}{2}$$

$$\ln L=\frac{1}{2}$$

$$L=e^{\frac{1}{2}}=\sqrt{e}$$

Example 5 Evaluate $\lim_{x\to0^+}(x^2+2^x)^{\cot x}$ 20 January 2001 A

Solution

$$L=\lim_{x\to0^+}(x^2+2^x)^{\cot x}=1^{\infty}$$

$$\ln L=\lim_{x\to0^+}\cot x\,\ln\,(x^2+2^x)=\lim_{x\to0^+}\frac{\ln\,(x^2+2^x)}{\tan x}=\frac{0}{0}$$

$$\overset{LR}{=}\lim_{x\to0^+}\frac{\frac{2x+2^x\ln 2}{x^2+2^x}}{\sec^2 x}=\frac{\frac{0+\ln 2}{0+1}}{1}=\ln 2$$

$$L=e^{\ln 2}=2$$

Example 6 (2 pts) Find the following limits. $\lim_{x\to0^+}(\sec x+x)^{\frac{1}{x}}.$ 34 July 9, 2011

Solution

$$L=\lim_{x\to0^+}(\sec x+x)^{\frac{1}{x}}=1^{\infty}$$

$$\ln L=\lim_{x\to0^+}\frac{1}{x}\ln(\sec x+x)=\lim_{x\to0^+}\frac{\ln(\sec x+x)}{x}=\frac{0}{0}$$

$$\overset{LR}{=}\lim_{x\to0^+}\frac{\frac{\sec x\tan x+1}{\sec x+x}}{1}=\frac{0+1}{1+0}=1$$

$$L=e^1=e$$

Example 7

Evaluate $\lim_{x\to 0}\left(\frac{\sinh 2x}{\sin x}\right)^{\tanh x}$ 24 May 2005 A

Solution

$$L_1 = \lim_{x\to 0}\frac{\sinh 2x}{\sin x} = \frac{0}{0}$$

$$\overset{LR}{=} \lim_{x\to 0}\frac{2\cosh 2x}{\cosh x} = 2$$

$$L = \lim_{x\to 0}\left(\frac{\sinh 2x}{\sin x}\right)^{\tanh x} = 2^0 = 1$$

Example 8

Evaluate $\lim_{x\to 0^+}(\sin x)^{\frac{1}{\ln x}}$, 28 July 2006 A

Solution

$$L = \lim_{x\to 0^+}(\sin x)^{\frac{1}{\ln x}}$$

$$\ln L = \lim_{x\to 0^+}\frac{\ln\sin x}{\ln x} = \frac{-\infty}{-\infty}$$

$$\overset{LR}{=} \lim_{x\to 0}\frac{\frac{\cos x}{\sin x}}{\frac{1}{x}} = \lim_{x\to 0}\frac{x\cos x}{\sin x} = \frac{0}{0}$$

$$\overset{LR}{=} \lim_{x\to 0}\frac{\cos x - x\sin x}{\cos x} = \frac{1-0}{1} \; 1$$

$$\therefore L = e$$

Example 9

Evaluate $\lim_{x\to 0}\left[\exp\left(\frac{1}{x^2}\right)\right]^{(\cosh x-1)}$ 30 January 2008

Solution

$$L = \lim_{x\to 0}\left[e^{\frac{1}{x^2}}\right]^{(\cosh x-1)}$$

$$\ln L = \lim_{x\to 0}(\cosh x-1)\ln e^{\frac{1}{x^2}} = \lim_{x\to 0}(\cosh x-1)\cdot\frac{1}{x^2}\ln e = \lim_{x\to 0}\frac{\cosh x-1}{x^2} = \frac{0}{0}$$

$$\overset{LR}{=} \lim_{x\to 0}\frac{\sinh x}{2x} = \frac{0}{0}$$

$$\overset{LR}{=} \lim_{x\to 0}\frac{\cos x}{2} = \frac{1}{2}$$

$$L = e^{\frac{1}{2}} = \sqrt{e}$$

Example 10 Find the value of $a > 0$ for which $\lim_{x\to\infty}\left(\frac{ax+1}{ax-1}\right)^{x} = 9$ 34 August 2009 A

Solution

$$L = \lim_{x\to\infty}\left(\frac{ax+1}{ax-1}\right)^{x}$$

$$\ln L = \lim_{x\to\infty} x\ln\left(\frac{ax+1}{ax-1}\right) = \lim_{x\to\infty}\frac{\ln\left(\frac{ax+1}{ax-1}\right)}{\frac{1}{x}} = \frac{0}{0}$$

$$L.R \to \lim_{x\to\infty}\frac{\frac{ax-1}{ax+1}\cdot\frac{a(ax-1)-a(ax+1)}{(ax-1)^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty}\frac{\frac{a(ax-1-ax-1)}{(ax+1)(ax-1)}}{\frac{-1}{x^2}} = \lim_{x\to\infty}\frac{\frac{-2a}{(ax+1)(ax-1)}}{\frac{-1}{x^2}}$$

$$L.R \to \lim_{x\to\infty}\frac{2ax^2}{a^2x^2-1} = \frac{2a}{a^2} = \frac{2}{a}$$

$$\therefore L = e^{\frac{2}{a}} \quad\to\quad \therefore e^{\frac{2}{a}} = 9 \quad\to\quad \therefore \frac{2}{a} = \ln 9$$

$$a = \frac{2}{\ln 9} = \frac{1}{\ln 3}$$

Example 11 which the indeterminate form does the function $(3-2\tanh x)^{\sinh x}$ have as $x\to\infty$ and state why ?

35 December 2004 A

Solution

$\tanh x \to 1 \quad as \quad x\to\infty$

$\sinh x \to \infty \quad as \quad x\to\infty$

$(3-2\tanh x)^{\sinh x} \Rightarrow 1^{\infty} \quad as \quad x\to\infty$

Homework

1	Find	$\lim_{x-\infty}\left(\cos\frac{1}{x}\right)^x$	7 July 1997
2	Find	$\lim_{x\to\infty}(2^x+3^x)^{\frac{1}{x}}$	12 July 2000 A
3	Evaluate	$\lim_{x\to 0^+}(1+\csc x)^{\sin x}$	23 January 2005 A
4	Evaluate	$\lim_{x\to 0^+}(1-3x)^{\cot x}$	33 May 2004
5	Evaluate	$\lim_{x\to\infty}(1-e^{-x})^{e^x}$	3 August 1995
6	Evaluate	$\lim_{x\to 1^+}(\ln x)^{x-1}$	25 August 2005 A
7	Evaluate	$\lim_{x\to 0^+}(e^x-1)^{\frac{1}{\ln x}}$	29 January 2007 A
8	Evaluate	$\lim_{x\to\infty}\left(\frac{x+2}{x-1}\right)^x$.	31 August 2008 A
9	Find	$\lim_{x\to\infty}(e^x)^{e^{-x}}$	18 July 2005 A
10	Evaluate	$\lim_{x\to 1}\frac{\ln x^{\sqrt{x}}}{x^3-1}$	18 July 2005 A
11	Find the following limits	$\lim_{x\to\infty}(1+e^x)^{1/x}$	15 July 2003 A
12	Evaluate	$\lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)^{2x+1}$	23 November 2007 A
13	Evaluate	$\lim_{n\to\infty}\left(\frac{x}{x-3}\right)^x$	4 July 1996
14	Evaluate	$\lim_{n\to\infty}(3x+e^x)^{\frac{2}{x}}$	2 May 1995
15	Evaluate	$\lim_{x\to\frac{\pi}{2}^-}(\tan x)^{\cos x}$	11 December 1997

Homework

16	Evaluate	$\lim_{x\to 1^+} (\ln x)^{\ln x}$	12 December 1997
17	Evaluate	$\lim_{x\to\infty} (1+2x)^{e^{-x}}$	13 May 1998
18	Evaluate	$\lim_{x\to\infty} (x+e^x)^{\frac{2}{x}}$	25 December 2001
19	Evaluate	$\lim_{x\to\infty} (1-e^{-2x})^x$	37 June 2005
20	Evaluate	$\lim_{x\to\infty} \left(\frac{x-3}{x+5}\right)^x$.	39 December 2005
21	Evaluate	$\lim_{x\to 0} \left(\frac{5^x+2^{x+1}}{3}\right)^{\frac{1}{x}}$	40 May 2006
22	Evaluate	$\lim_{x\to 0^+} x^{\tan x}$	14 November 1998
23	Evaluate	$\lim_{x\to\infty} (1-e^x)^{e^{-x}}$	15 December 1998
24	Evaluate	$\lim_{x\to 0^+} (1+x)^{\frac{1}{\tan^{-1} x}}$	4 December 1995
25	(3pts) Evaluate the limit , if it exists	$\lim_{x\to 1^+} \left(\frac{4}{\pi}\tan^{-1} x\right)^{1/(\ln x)}$.	30 April 11, 2010
26	Evaluate the following.	$\lim_{x\to 1}(1+\ln x)^{1/(x-1)}$. [2 mark]	31 10 July 2010
27	(3 pts). Find the limit if it exists :	$\lim_{x\to\infty} (\cosh x)^{1/x}$	33 April 10, 2011
28	(2 points) Evaluate the following limit	$\lim_{x\to 1^+} (\ln x)^{\ln x}$	50 Dec. 15, 2009
29	(3 pts.) Find the following limits, if it exist.	$\lim_{x\to 0}(1-\cos x)^{1-\sec x}$	53 11 Dec. 2010
30	(4 pts.) Evaluate	$\lim_{x\to\infty} (e^{2x}+3x)^{5/x}$	36 June 6, 2010

Homework

31	Evaluate	$\lim_{x \to 0} \lvert x \rvert^{\sin x}$	22 June 2004 A
32	Evaluate	$\lim_{x \to \infty} \left(\coth \frac{x}{2} \right)^{x}$	
33	Find	$\lim_{x \to 1} (2 - x)^{\tan(\pi x/2)}$	41 14 January 2012

31 Evaluate $\lim_{x \to 0} |x|^{\sin x}$ 22 June 2004 A

Solution

$x > 0$

$L_1 = \lim_{x \to 0} |x|^{\sin x} = 0^0$

$\ln L_1 = \lim_{x \to 0^+} \sin x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\csc x} = \frac{-\infty}{\infty}$

$\overset{LR}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{\csc x \cot x} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{1}{\sin x} . \frac{\cos x}{\sin x}} = \lim_{x \to 0^+} \frac{\sin^2 x}{x \cos x} = \frac{0}{0}$

$\overset{LR}{=} \lim_{x \to 0} \frac{2 \sin x \cos x}{\cos x - x \sin x} = \frac{0}{1} = 0$

$L_1 = e^0 = 1$

$x < 0$

$L_2 = \lim_{x \to 0^-} (-x)^{\sin x}$

$\ln L_2 = \lim_{x \to 0^-} \sin x \ \ln(-x) = \lim_{x \to 0^-} \frac{\ln(-x)}{\sin x} = \frac{-\infty}{\infty}$

$\overset{LR}{=} \lim_{x \to 0^-} \frac{\frac{-1}{-x}}{\csc x \cot x} = \lim_{x \to 0^-} \frac{\frac{1}{x}}{\csc x \cot x} = 0$

$L_2 = e^0 = 1$

$\therefore L = 1$

32 Evaluate $\lim_{x \to \infty} \left(\coth \frac{x}{2}\right)^x$

Solution

$L = \lim_{x \to \infty} \left[\coth\left(\frac{x}{2}\right)\right]^x = 1^\infty$

$\ln L = \lim_{x \to \infty} x \ln\left[\coth\left(\frac{x}{2}\right)\right] = \lim_{x \to \infty} \frac{\ln\left[\coth\left(\frac{x}{2}\right)\right]}{\frac{1}{x}} = \frac{0}{0}$

$\overset{LR}{=} \lim_{x \to \infty} \frac{\frac{-\frac{1}{2}\operatorname{csch}^2\left(\frac{x}{2}\right)}{\coth\left(\frac{x}{2}\right)}}{\frac{-1}{x^2}} = \frac{x^2 \operatorname{csch}^2\left(\frac{x}{2}\right)}{2 \coth\left(\frac{x}{2}\right)} = = \lim_{x \to \infty} \frac{x^2}{2 \sinh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{2}\right)} = \lim_{x \to \infty} \frac{x^2}{\sinh x} = \frac{\infty}{\infty}$

$\overset{LR}{=} \lim_{x \to \infty} \frac{2x}{\cosh x} = \frac{\infty}{\infty}$

$\overset{LR}{=} \lim_{x \to \infty} \frac{2}{\sinh x} = 0$

$L = e^0 = 1$